SUT Journal of Mathematics
Vol. 33, No. 2 (1997), 207–237

GENTZEN-TYPE FORMULATION OF SOME MODAL LOGICS

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(Received October 13, 1997)

Abstract. In this paper, we investigate what kind of modal logics can be constructed into the Gentzen-type formulation which enjoys the Cut elimination theorem. We treat *normal*, *regular* and *monotonic* modal logics. A normal modal logic is the modal logic with the Rule of Necessitation and the axiom K: $\Box(A \supset B) \supset (\Box A \supset \Box B)$, a regular modal logic with the rule of inference RR : $\frac{A,\Gamma\to B}{\Box A,\Box\Gamma\to\Box B}$ and a monotonic modal logic with the rule of inference RM : $\frac{A\to B}{\Box A\to\Box B}$. We provide Gentzen-type formulation and prove the cut elimination theorem for the following modal logics: RT4 (a regular modal logic with the axioms T: $\Box A \supset A$ and 4: $\Box A \supset \Box\Box A$), MT4 (a monotonic modal logic with the axioms T and 4), MNT4 (a monotonic modal logic with the Rule of Necessitation and the axioms T and 4), RT (a regular modal logic with the axiom T), MT (a monotonic modal logic with the axiom T), MNT (a monotonic modal logic with the Rule of Necessitation and the axiom T), KL (a normal modal logic with the axiom L: $\Box A \supset (A \vee \Box B)$) and RTA (a regular modal logic with axioms T and A: $\Box A \& B \supset \Box B$).

AMS 1991 *Mathematics Subject Classification.* Primary 03B45.

Key words and phrases. Gentzen-type formulation, sequent calculus, normal modal logic, regular modal logic, monotonic modal logic.

§1. Introduction

By **LK**, we mean the classical propositional logic provided with the four usual logical operators $\supset$, &, $\vee$, $\neg$ and constructed as in Gentzen [3]. Concerning the English terminologies for Gentzen-type formulation and the proof of the Cut elimination theorem, we mainly follow Kleene [6], with which, therefore, we assume the familiarity.

In order to construct *modal logics*, we add, as usual, one modal operator $\Box$ (*necessity*) to **LK**. The definition of well-formed formulas is just as usual and we use upper case Latin letters $A, B, C, \cdots$ for formulas. Also we use upper case Greek letters $\Delta, \Gamma, \Pi, \Sigma$ for finite, *possibly empty*, sequences of formulas. For a sequence Γ of formulas, the expression $\Box\Gamma$ means the sequence of formulas formed by prefixing the symbol $\Box$ in front of each formula of Γ. If Γ is empty, then $\Box\Gamma$ is also empty.

As usual, an expression of the form $\Gamma \to \Delta$ is called *sequent*. An additional axiom A for Gentzen-type formulation means that we can use the sequent $\to A$ as a beginning sequent.

For the definitions of *normal*, *regular* and *monotonic* modal logics, we cite Chellas [1] as follows:

Definition 1.1. (1) A *normal* modal logic is a modal logic with the additional axiom

$$\mathrm{K} : \Box(A \supset B) \supset (\Box A \supset \Box B)$$

and the Rule of Necessitation

$$\mathrm{RN} : \quad \frac{\to A}{\to \Box A}$$

The *smallest normal logic* is called as $\mathbf{K}$, which is expressed, as usual, as

$$\mathbf{K} = \mathbf{LK} + \mathrm{K} + \mathrm{RN}.$$

(2) A *regular* modal logic is a modal logic with the additional rule of inference

$$\mathrm{RR} : \quad \frac{A, \Gamma \to B}{\Box A, \Box\Gamma \to \Box B}$$

The *smallest regular logic* is called as $\mathbf{R}$, that is, $\mathbf{R} = \mathbf{LK} + \mathrm{RR}$.

(3) A *monotonic* modal logic is a modal logic with the additional rule of inference

$$\mathrm{RM} : \quad \frac{A \to B}{\Box A \to \Box B}$$

The *smallest monotonic logic* is called as $\mathbf{M}$, that is, $\mathbf{M} = \mathbf{LK} + \mathrm{RM}$.

Now we introduce the modal logics which we investigate in this paper.

Definition 1.2. (1) The logic **RT4** is a regular modal logic with the additional axioms $\mathrm{T} : \Box A \supset A$ and $4 : \Box A \supset \Box\Box A$, that is, $\mathbf{RT4} = \mathbf{R}+\mathrm{T}+4$.

(2) $\mathbf{MT4}$=$\mathbf{M}$+T+4.

(3) $\mathbf{MNT4}$=$\mathbf{M}$+RN+T+4.

(4) **S4**=**K**+T+4.

(5) **RT**=**R**+T.

(6) **MT**=**M**+T.

(7) **MNT**=**M**+RN+T.

(8) **KT**=**K**+T.

(9) **KL**=**K**+L, where L is the axiom $\Box A \supset (A \vee \Box B)$.

(10) **KD**=**K**+D, where D is the axiom $\Box A \supset \neg\Box\neg A$.

(11) **RTA**=**R**+T+A, where A is the axiom $\Box A \& B \supset \Box B$.

(12) **S5**=**K**+T+5, where 5 is the axiom $\Box A \vee \Box\neg\Box A$.

Figure 1

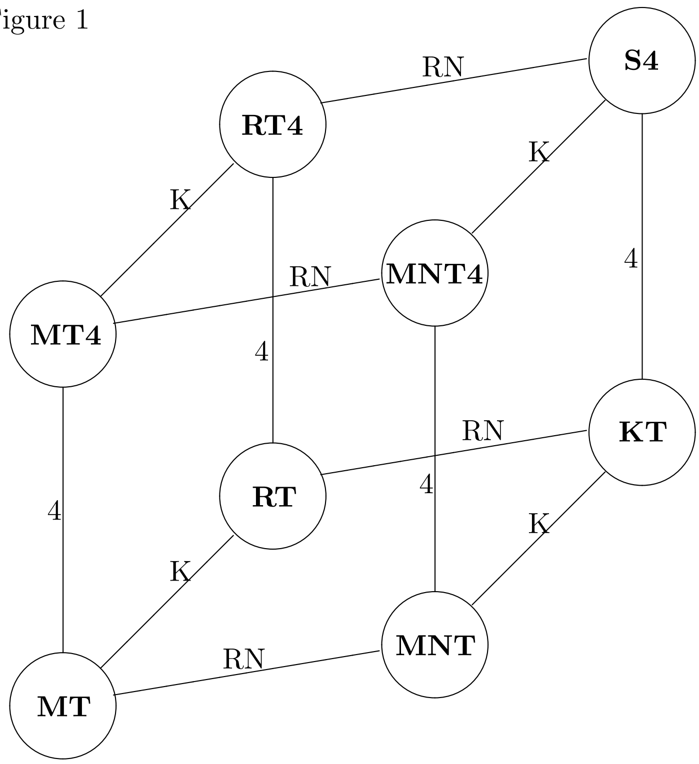
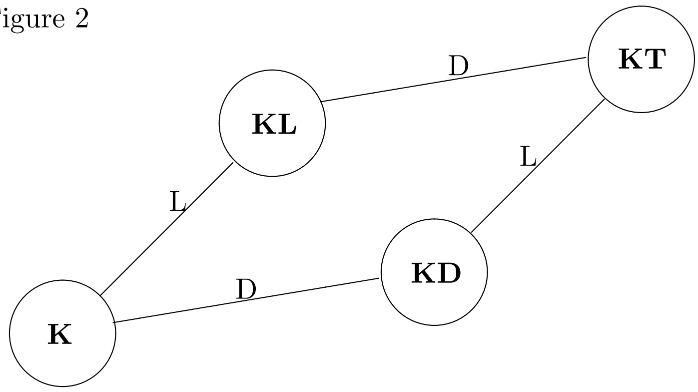


Figure 2



The above Figures 1 and 2 illustrate the inclusion relations among these logics. There, the lines indicate the inclusion relations, that is, extensions of logics are reached by going upward along the lines. The inclusions in the Figure 1 are proved in §10. The inclusions in the Figure 2 are well known.[1]

For eight of these logics, we provide Gentzen-type formulation and prove the Cut elimination theorems[2] .

§2. Gentzen-type Formulation RT4* of RT4

Now we reconstruct **RT4** into Gentzen-type formulation **RT4*** and prove the Cut elimination theorem for **RT4***.

Definition 2.1. The modal logic **RT4*** is constructed from **LK** by adding the modal operator $\Box$ and the rules of inference of the forms:

$$\text{LB}: \quad \frac{A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}$$

and

$$\text{RT4}: \quad \frac{\Box B, \Box\Sigma \to A}{\Box B, \Box\Sigma \to \Box A}$$

that is, **RT4*** = **LK** +LB + RT4.

Next, we prove that **RT4**=**RT4*** as logics, that is, the sets of provable formulas of **RT4** and **RT4*** are the same.

(1)**RT4**$\supseteq$**RT4***, as sets of provable formulas.[3]

[1] See eg. Lemmon [2], where the axiom L is given in the form: $\Diamond\top \supset (\Box A \supset A)$.

[2] The logics **S4** and **KT** are treated in the pioneer work by Ohnishi and Matsumoto [4]. (There, **KT** is treated by the name **M**.)

[3] We use the symbol $\supseteq$ to mean the set-inclusion, not necessarily the strict inclusion. Later we use $\supset$ to mean the strict inclusion.

For this, we only need to prove the following two lemmas:

Lemma 2.2. *The inference* LB *is permissible in* **RT4**.

Proof.

$$\dfrac{\Box A \to A \quad A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}\ Cut$$

■

Lemma 2.3. *The inference* RT4 *is permissible in* **RT4**.

Proof.

$$\dfrac{\dfrac{\Box B \to \Box\Box B \quad \dfrac{\Box B, \Box C_1, \cdots, \Box C_n \to A}{\Box\Box B, \Box\Box C_1, \cdots, \Box\Box C_n \to \Box A}\ \mathrm{RR}}{\Box B, \Box\Box C_1, \cdots, \Box\Box C_n \to \Box A}\ Cut}{\dfrac{(Cut, Interchange)}{\Box B, \Box C_1, \cdots, \Box C_n \to \Box A}}$$

where $n \geq 0$. ■

(2)**RT4*** $\supseteq$**RT4**, as sets of provable formulas.

For this, we only need to prove the following three lemmas, the first two of which are almost immediate:

Lemma 2.4. *The axiom* T: $\Box A \supset A$ *is provable in* **RT4***.

Lemma 2.5. *The axiom* 4: $\Box A \supset \Box\Box A$ *is provable in* **RT4***.

Lemma 2.6. *The inference* RR *is permissible in* **RT4***.

Proof.

$$\dfrac{\dfrac{\dfrac{A_1, \cdots, A_n \to B}{\Box A_1, A_2, \cdots, A_n \to B}\ \mathrm{LB}}{\dfrac{(\mathrm{LB}, Interchange)}{\Box A_1, \cdots, \Box A_n \to B}}}{\Box A_1, \cdots, \Box A_n \to \Box B}\ \mathrm{RT4}$$

where $n \geq 1$. ■

So we have the following:

Theorem 2.7. **RT4=RT4***.

Thus, the sets of provable formulas of **RT4** and **RT4*** are the same.

Now we prove the Cut elimination theorem for **RT4***.

Theorem 2.8. (Cut elimination) *Any* **RT4*** *proof can be transformed into an* **RT4*** *proof of the same endsequent without use of the inference Cut.*

As usual, we introduce an additional inference *Mix* as follows:

$$\frac{\Gamma \to \Delta \quad \Sigma \to \Pi}{\Gamma, \Sigma_A \to \Delta_A, \Pi}\ Mix(A)$$

where Δ_A and Σ_A mean the sequences of formulas obtained from Δ and Σ by eliminating the *Mix formula* A from them. And we replace the inference *Cut* with *Mix*. So the *Cut elimination* means the elimination of *Mix*.

Our proof proceeds along the line of Kleene [6] by the induction on *rank* and *grade* of *Mix* which produces the endsequent of the proof and which is the only one *Mix* used in the proof. For the definition of *rank* and *grade* and for the details of the usual Cut elimination method, we refer to Kleene [6].

Hereafter, S_1 means the left upper sequent of the inference *Mix*, and S_2 the right upper sequent of the inference *Mix*:

$$\frac{S_1 \quad S_2}{S}\ Mix$$

where S is the endsequent.

As we follow Kleene [6], it is obvious that we only need to treat the following three cases:

(1) The case where *the left rank* $= 1$, *the right rank* $= 1$, and the outermost symbol of the *Mix formula* is $\Box$.

(2) The case where *the right rank* > 1, *the left rank* $= 1$, and S_2 is derived by LB or RT4.

(3) The case where *the left rank* > 1, and S_1 is derived by LB or RT4.

Proof. (1) The case where *the left rank* $= 1$, *the right rank* $= 1$, and the outermost symbol of the *Mix formula* is $\Box$.

In this case, S_1 is derived by RT4 and S_2 is derived by LB. So, the bottom part of the proof is as follows:

$$\dfrac{\dfrac{\Box A, \Box\Gamma \to B}{\Box A, \Box\Gamma \to \Box B}\ \mathrm{RT4} \quad \dfrac{B, \Sigma \to \Pi}{\Box B, \Sigma \to \Pi}\ \mathrm{LB}}{\Box A, \Box\Gamma, \Sigma \to \Pi}\ Mix(\Box B)$$

where $\Box B$ does not appear in Σ, since *the right rank* $= 1$.
We transform the above to the following:

$$\dfrac{\dfrac{\Box A, \Box\Gamma \to B \quad B, \Sigma \to \Pi}{\Box A, \Box\Gamma, \Sigma_B \to \Pi}\, Mix(B)}{\dfrac{(Thinning, Interchange)}{\Box A, \Box\Gamma, \Sigma \to \Pi}}$$

In the transformed figure, the *grade* of the *Mix* gets smaller. Hence by the hypothesis of the induction, we can obtain a proof without *Mix* of the original endsequent.

(2) The case where *the right rank* > 1, *the left rank* $= 1$, and S_2 is derived by LB or RT4.

(A) The case where S_2 is derived by LB. Here, we divide the case into (a) the subcase where the *Mix formula* is $\Box A$ and (b) the subcase where the *Mix formula* is not $\Box A$.

(a) When the *Mix formula* is $\Box A$, the bottom part of the proof is as follows:

$$\dfrac{\Gamma \to \Delta \quad \dfrac{A, \Sigma \to \Pi}{\Box A, \Sigma \to \Pi}\,\text{LB}}{\Gamma, \Sigma_{\Box A} \to \Delta_{\Box A}, \Pi}\, Mix(\Box A)$$

We transform the above to the following:

$$\dfrac{\dfrac{\Gamma \to \Delta \quad \dfrac{\dfrac{\dfrac{\Gamma \to \Delta \quad A, \Sigma \to \Pi}{\Gamma, A, \Sigma_{\Box A} \to \Delta_{\Box A}, \Pi}\, Mix_1(\Box A)}{\dfrac{(Interchange)}{A, \Gamma, \Sigma_{\Box A} \to \Delta_{\Box A}, \Pi}}}{\Box A, \Gamma, \Sigma_{\Box A} \to \Delta_{\Box A}, \Pi}\,\text{LB}}{\Gamma, \Gamma, \Sigma_{\Box A} \to \Delta_{\Box A}, \Delta_{\Box A}, \Pi}\, Mix_2(\Box A)}{\dfrac{(Contraction, Interchange)}{\Gamma, \Sigma_{\Box A} \to \Delta_{\Box A}, \Pi}}$$

In the transformed figure, *Mix* is used twice. The *rank* of Mix_1 gets smaller than that of the original. So, Mix_1 can be eliminated. After the elimination of Mix_1, Mix_2 can be eliminated since the *right rank* of Mix_2 is only 1, while the *left rank* of Mix_2 is the same as in the original. So, we can obtain a proof without *Mix* of the original endsequent.

(b) When the *Mix formula* is not $\Box A$, the bottom part of the proof is as follows:

$$\dfrac{\Gamma \to \Delta \quad \dfrac{A, \Sigma \to \Pi}{\Box A, \Sigma \to \Pi}\,\text{LB}}{\Gamma, \Box A, \Sigma_D \to \Delta_D, \Pi}\, Mix(D)$$

We transform the above to the following:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \to \Delta \quad A, \Sigma \to \Pi}{\Gamma, (A)_D, \Sigma_D \to \Delta_D, \Pi}\, Mix(D)}{(Thinning, Interchange)}}{A, \Gamma, \Sigma_D \to \Delta_D, \Pi}}{\Box A, \Gamma, \Sigma_D \to \Delta_D, \Pi}\, \mathrm{LB}}{\dfrac{(Interchange)}{\Gamma, \Box A, \Sigma_D \to \Delta_D, \Pi}}$$

In the transformed figure, the *rank* of the *Mix* gets smaller. Hence, we can obtain a proof without *Mix* of the original endsequent.

(B) When S_2 is derived by RT4, the bottom part of the proof is as follows:

$$\dfrac{\dfrac{\Box C, \Box\Gamma \to D}{\Box C, \Box\Gamma \to \Box D}\, \mathrm{RT4} \quad \dfrac{\Box A, \Box\Sigma \to B}{\Box A, \Box\Sigma \to \Box B}\, \mathrm{RT4}}{\Box C, \Box\Gamma, (\Box A)_{\Box D}, (\Box\Sigma)_{\Box D} \to \Box B}\, Mix(\Box D)$$

where S_1 is derived by RT4, since the outermost symbol of the *Mix formula* is $\Box$ and *the left rank* $= 1$.
We transform the above to the following:

$$\dfrac{\dfrac{\Box C, \Box\Gamma \to \Box D \quad \Box A, \Box\Sigma \to B}{\Box C, \Box\Gamma, (\Box A)_{\Box D}, (\Box\Sigma)_{\Box D} \to B}\, Mix(\Box D)}{\Box C, \Box\Gamma, (\Box A)_{\Box D}, (\Box\Sigma)_{\Box D} \to \Box B}\, \mathrm{RT4}$$

In the transformed figure, the *rank* of the *Mix* gets smaller. Hence, we can obtain a proof without *Mix* of the original endsequent.

(3) The case where *the left rank* > 1, and S_1 is derived by LB or RT4.

In this case, S_1 cannot be derived by RT4 since *the left rank* > 1. So we treat only the case where S_1 is derived by LB.

When S_1 is derived by LB, the bottom part of the proof is as follows:

$$\dfrac{\dfrac{A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}\, \mathrm{LB} \quad \Sigma \to \Pi}{\Box A, \Gamma, \Sigma_D \to \Delta_D, \Pi}\, Mix(D)$$

We transform the above to the following:

$$\dfrac{\dfrac{A, \Gamma \to \Delta \quad \Sigma \to \Pi}{A, \Gamma, \Sigma_D \to \Delta_D, \Pi}\, Mix(D)}{\Box A, \Gamma, \Sigma_D \to \Delta_D, \Pi}\, \mathrm{LB}$$

In the transformed figure, the *rank* of the *Mix* gets smaller. Hence, we can obtain a proof without *Mix* of the original endsequent.

■

§3. Gentzen-type Formulation MT4* of MT4

Now we reconstruct **MT4** into a Gentzen-type formulation **MT4*** and prove the Cut elimination theorem for **MT4***.

Definition 3.1. The modal logic **MT4*** is constructed from **LK** by adding the modal operator $\Box$ and the rules of inference LB and

$$\mathrm{MT4}: \quad \frac{\Box A \to B}{\Box A \to \Box B}$$

that is, **MT4*** = **LK** + LB + MT4.

Next, we prove that **MT4**=**MT4*** as logics.

(1)**MT4**$\supseteq$**MT4***.

For this, we only need to prove the following two lemmas, the first one of which is immediate:

Lemma 3.2. *The inference* LB *is permissible in* **MT4**.

Lemma 3.3. *The inference* MT4 *is permissible in* **MT4**.

Proof.

$$\dfrac{\Box A \to \Box\Box A \qquad \dfrac{\Box A \to B}{\Box\Box A \to \Box B}\,\mathrm{RM}}{\Box A \to \Box B}\,\mathit{Cut}$$

■

(2)**MT4*** $\supseteq$**MT4**.

For this, we only need to prove the following three lemmas, which are almost immediate:

Lemma 3.4. *The axiom* T: $\Box A \supset A$ *is provable in* **MT4***.

Lemma 3.5. *The axiom* 4: $\Box A \supset \Box\Box A$ *is provable in* **MT4***.

Lemma 3.6. *The inference* RM *is permissible in* **MT4***.

So we have the following:

Theorem 3.7. **MT4**=**MT4***.

Now we prove the Cut elimination theorem for **MT4***.

Theorem 3.8. (Cut elimination) *Any* **MT4*** *proof can be transformed into an* **MT4*** *proof of the same endsequent without use of the inference Cut.*

As before, it is obvious that we only need to treat the following three cases:

(1) The case where *the left rank* $= 1$, *the right rank* $= 1$, and the outermost symbol of the *Mix formula* is $\Box$.

(2) The case where *the right rank* > 1, *the left rank* $= 1$, and S_2 is derived by LB or MT4.

(3) The case where *the left rank* > 1, and S_1 is derived by LB or MT4.

Proof. (1) The case where *the left rank* $= 1$, *the right rank* $= 1$, and the outermost symbol of the *Mix formula* is $\Box$.

In this case, S_1 is derived by MT4 and S_2 is derived by LB. So, the bottom part of the proof is as follows:

$$\dfrac{\dfrac{\Box A \to B}{\Box A \to \Box B}\,\text{MT4} \qquad \dfrac{B, \Sigma \to \Pi}{\Box B, \Sigma \to \Pi}\,\text{LB}}{\Box A, \Sigma \to \Pi}\,Mix(\Box B)$$

where $\Box B$ does not appear in Σ, since *the right rank* $= 1$.
We transform the above to the following:

$$\dfrac{\dfrac{\dfrac{\Box A \to B \quad B, \Sigma \to \Pi}{\Box A, \Sigma_B \to \Pi}\,Mix(B)}{(Thinning, Interchange)}}{\Box A, \Sigma \to \Pi}$$

In the transformed figure, the *grade* of the *Mix* gets smaller. Hence, we can obtain a proof without *Mix* of the original endsequent.

(2) The case where *the right rank* > 1, *the left rank* $= 1$, and S_2 is derived by LB or MT4.

(A) The case where S_2 is derived by LB can be treated as in the proof of Theorem 2.8(2)(A).

(B) When S_2 is derived by MT4, the bottom part of the proof is as follows:

$$\dfrac{\dfrac{\Box C \to A}{\Box C \to \Box A}\,\text{MT4} \qquad \dfrac{\Box A \to B}{\Box A \to \Box B}\,\text{MT4}}{\Box C \to \Box B}\,Mix(\Box A)$$

where S_1 is derived by MT4, since the outermost symbol of the *Mix formula* is $\Box$ and *the left rank* $= 1$.

We transform the above to the following:

$$\dfrac{\dfrac{\Box C \to \Box A \quad \Box A \to B}{\Box C \to B}\,Mix(\Box A)}{\Box C \to \Box B}\,\text{MT4}$$

In the transformed figure, the *rank* of the *Mix* gets smaller. Hence, we can obtain a proof without *Mix* of the original endsequent.

(3) The case where *the left rank* > 1, and S_1 is derived by LB or MT4.

In this case, S_1 cannot be derived by MT4 since *the left rank* > 1. The case where S_1 is derived by LB can be treated as in the proof of Theorem 2.8(3).

■

§4. Gentzen-type Formulation MNT4* of MNT4

Now we reconstruct **MNT4** into a Gentzen-type formulation **MNT4*** and prove the Cut elimination theorem for **MNT4***.

Definition 4.1. The modal logic **MNT4*** is constructed from **LK** by adding the modal operator $\Box$ and the rules of inference LB, RN and MT4, that is, **MNT4*** = **LK** + LB + RN + MT4.

Next, we prove that **MNT4**=**MNT4*** as logics.

(1)**MNT4**$\supseteq$**MNT4***.

For this, we only need to prove the following two lemmas, which are almost immediate:

Lemma 4.2. *The inference* LB *is permissible in* **MNT4**.

Lemma 4.3. *The inference* MT4 *is permissible in* **MNT4**.

(2)**MNT4*** $\supseteq$**MNT4**.

For this, we only need to prove the following three lemmas, which are almost immediate:

Lemma 4.4. *The axiom* T: $\Box A \supset A$ *is provable in* **MNT4***.

Lemma 4.5. *The axiom* 4: $\Box A \supset \Box\Box A$ *is provable in* **MNT4***.

Lemma 4.6. *The inference* RM *is permissible in* **MNT4***.

So we have the following:

Theorem 4.7. **MNT4=MNT4***.

Now we prove the Cut elimination theorem for **MNT4***.

Theorem 4.8. (Cut elimination) *Any* **MNT4*** *proof can be transformed into an* **MNT4*** *proof of the same endsequent without use of the inference Cut.*

As before, it is obvious that we only need to treat the following three cases:

(1) The case where *the left rank* $= 1$, *the right rank* $= 1$, and the outermost symbol of the *Mix formula* is $\Box$.

(2) The case where *the right rank* > 1, *the left rank* $= 1$, and S_2 is derived by LB, RN or MT4.

(3) The case where *the left rank* > 1, and S_1 is derived by LB, RN or MT4.

Proof. (1) The case where *the left rank* $= 1$, *the right rank* $= 1$, and the outermost symbol of the *Mix formula* is $\Box$. In this case, we further divide the case into (A) the subcase where S_1 is derived by RN and S_2 is derived by LB, and (B) the subcase where S_1 is derived by MT4 and S_2 is derived by LB.

(A) When S_1 is derived by RN and S_2 is derived by LB, the bottom part of the proof is as follows:

$$\dfrac{\dfrac{\to A}{\to \Box A}\ \text{RN} \qquad \dfrac{A, \Sigma \to \Pi}{\Box A, \Sigma \to \Pi}\ \text{LB}}{\Sigma \to \Pi}\ Mix(\Box A)$$

where $\Box A$ does not appear in Σ, since *the right rank* $= 1$.
We transform the above to the following:

$$\dfrac{\dfrac{\to A \quad A, \Sigma \to \Pi}{\Sigma_A \to \Pi}\ Mix(A)}{\dfrac{(Thinning, Interchange)}{\Sigma \to \Pi}}$$

In the transformed figure, the *grade* of the *Mix* gets smaller. Hence, we can obtain a proof without *Mix* of the original endsequent.

(B) The case where S_1 is derived by MT4 and S_2 is derived by LB can be treated as in the proof of Theorem 3.8(1).

(2) The case where *the right rank* > 1, *the left rank* $= 1$, and S_2 is derived by LB, RN or MT4.

In this case, S_2 cannot be derived by RN since *the right rank* > 1. So we treat only the case where S_2 is derived by LB or MT4.

(A) The case where S_2 is derived by LB can be treated as in the proof of Theorem 2.8(2)(A).

(B) The case where S_2 is derived by MT4. Since the outermost symbol of the *Mix formula* is $\Box$ and *the left rank* $= 1$, S_1 is derived by RN or MT4.

(a) When S_1 is derived by RN, the bottom part of the proof is as follows:

$$\dfrac{\dfrac{\to A}{\to \Box A}\ \text{RN} \qquad \dfrac{\Box A \to B}{\Box A \to \Box B}\ \text{MT4}}{\to \Box B}\ Mix(\Box A)$$

We transform the above to the following:

$$\dfrac{\dfrac{\to \Box A \quad \Box A \to B}{\to B}\ Mix(\Box A)}{\to \Box B}\ \text{RN}$$

In the transformed figure, the *rank* of the *Mix* gets smaller. Hence, we can obtain a proof without *Mix* of the original endsequent.

(b) The case where S_1 is derived by MT4 can be treated as in the proof of Theorem 3.8(2)(B).

(3) The case where *the left rank* > 1, and S_1 is derived by LB, RN or MT4.

In this case, S_1 cannot be derived by RN nor by MT4 since *the left rank* > 1. The case where S_1 is derived by LB can be treated as in the proof of Theorem 2.8(3).

■

§5. Gentzen-type Formulation RT* of RT

Now we reconstruct **RT** into a Gentzen-type formulation **RT*** and prove the Cut elimination theorem for **RT***.

Definition 5.1. The modal logic **RT*** is constructed from **LK** by adding the modal operator $\Box$ and the rules of inference LB and RR, that is, **RT*** = **LK** + LB + RR.

Next, we prove that **RT**=**RT*** as logics.

(1)**RT**$\supseteq$**RT***.

For this, we only need to prove the following, which is immediate:

Lemma 5.2. *The inference* LB *is permissible in* **RT**.

(2)**RT*** $\supseteq$**RT**.

For this, we only need to prove the following, which is immediate:

Lemma 5.3. *The axiom* T: $\Box A \supset A$ *is provable in* **RT***.

So we have the following:

Theorem 5.4. **RT**=**RT***.

The following Cut elimination theorem for **RT*** has been proved in Ohnishi and Matsumoto [4] as the logic **Q2**.

Theorem 5.5. (Cut elimination) *Any* **RT*** *proof can be transformed into an* **RT*** *proof of the same endsequent without use of the inference Cut.*

§6. Gentzen-type Formulation $\mathbf{MT}^*$ of MT

Now we reconstruct **MT** into a Gentzen-type formulation $\mathbf{MT}^*$ and prove the Cut elimination theorem for $\mathbf{MT}^*$.

Definition 6.1. The modal logic $\mathbf{MT}^*$ is constructed from **LK** by adding the modal operator $\Box$ and the rules of inference LB and RM, that is, $\mathbf{MT}^* = \mathbf{LK} + \mathrm{LB} + \mathrm{RM}$.

Next, we prove that $\mathbf{MT}=\mathbf{MT}^*$ as logics.

(1)$\mathbf{MT}\supseteq\mathbf{MT}^*$.

For this, we only need to prove the following, which is immediate:

Lemma 6.2. *The inference* LB *is permissible in* **MT**.

(2)$\mathbf{MT}^* \supseteq\mathbf{MT}$, as sets of provable formulas.

For this, we only need to prove the following, which is immediate:

Lemma 6.3. *The axiom* T: $\Box A \supset A$ *is provable in* $\mathbf{MT}^*$.

So we have the following:

Theorem 6.4. $\mathbf{MT}=\mathbf{MT}^*$.

Now we prove the Cut elimination theorem for $\mathbf{MT}^*$.

Theorem 6.5. (Cut elimination) *Any* $\mathbf{MT}^*$ *proof can be transformed into an* $\mathbf{MT}^*$ *proof of the same endsequent without use of the inference Cut.*

As before, it is obvious that we only need to treat the following three cases:

(1) The case where *the left rank* $= 1$, *the right rank* $= 1$, and the outermost symbol of the *Mix formula* is $\Box$.

(2) The case where *the right rank* > 1, and S_2 is derived by LB or RM.

(3) The case where *the left rank* > 1, and S_1 is derived by LB or RM.

Proof. (1) The case where *the left rank* $= 1$, *the right rank* $= 1$, and the outermost symbol of the *Mix formula* is $\Box$.

In this case, we further divide the case into (A) the subcase where S_1 is derived by RM and S_2 is derived by RM, and (B) the subcase where S_1 is derived by RM and S_2 is derived by LB.

(A) When S_1 is derived by RM and S_2 is derived by RM, the bottom part of the proof is as follows:

$$\dfrac{\dfrac{A \to B}{\Box A \to \Box B}\,\mathrm{RM} \quad \dfrac{B \to C}{\Box B \to \Box C}\,\mathrm{RM}}{\Box A \to \Box C}\,Mix(\Box B)$$

We transform the above to the following:

$$\dfrac{\dfrac{A \to B \quad B \to C}{A \to C}\,Mix(B)}{\Box A \to \Box C}\,\mathrm{RM}$$

In the transformed figure, the *grade* of the *Mix* gets smaller. Hence, we can obtain a proof without *Mix* of the original endsequent.

(B) When S_1 is derived by RM and S_2 is derived by LB, the bottom part of the proof is as follows:

$$\dfrac{\dfrac{A \to B}{\Box A \to \Box B}\,\mathrm{RM} \quad \dfrac{B, \Sigma \to \Pi}{\Box B, \Sigma \to \Pi}\,\mathrm{LB}}{\Box A, \Sigma \to \Pi}\,Mix(\Box B)$$

where $\Box B$ does not appear in Σ, since *the right rank* $= 1$.
We transform the above to the following:

$$\dfrac{\dfrac{\dfrac{A \to B \quad B, \Sigma \to \Pi}{A, \Sigma_B \to \Pi}\,Mix(B)}{A, \Sigma \to \Pi}\,(Thinning, Interchange)}{\Box A, \Sigma \to \Pi}\,\mathrm{LB}$$

In the transformed figure, the *grade* of the *Mix* gets smaller. Hence, we can obtain a proof without *Mix* of the original endsequent.

(2) The case where *the right rank* > 1, and S_2 is derived by LB or RM.

In this case, S_2 cannot be derived by RM since *the right rank* > 1. The case where S_2 is derived by LB can be treated as in the proof of Theorem 2.8(2)(A).

(3) The case where *the left rank* > 1, and S_1 is derived by LB or RM.

In this case, S_1 cannot be derived by RM since *the left rank* > 1. The case where S_1 is derived by LB can be treated as in the proof of Theorem 2.8(3).

■

§7. Gentzen-type Formulation MNT* of MNT

Now we reconstruct **MNT** into a Gentzen-type formulation **MNT*** and prove the Cut elimination theorem for **MNT***.

Definition 7.1. The modal logic **MNT*** is constructed from **LK** by adding the modal operator $\Box$ and the rules of inference LB, RN and RM, that is, **MNT*** = **LK** + LB + RN +RM.

Next, we prove that **MNT**=**MNT*** as logics.

(1)**MNT**$\supseteq$**MNT***.

For this, we only need to prove the following, which is immediate:

Lemma 7.2. *The inference* LB *is permissible in* **MNT**.

(2)**MNT*** $\supseteq$**MNT**.

For this, we only need to prove the following, which is immediate:

Lemma 7.3. *The axiom* T: $\Box A \supset A$ *is provable in* **MNT***.

So we have the following:

Theorem 7.4. **MNT**=**MNT***.

Now we prove the Cut elimination theorem for **MNT***.

Theorem 7.5. (Cut elimination) *Any* $\mathbf{MNT}^*$ *proof can be transformed into an* $\mathbf{MNT}^*$ *proof of the same endsequent without use of the inference Cut.*

As before, it is obvious that we only need to treat the following three cases:

(1) The case where *the left rank* $= 1$, *the right rank* $= 1$, and the outermost symbol of the *Mix formula* is $\Box$.

(2) The case where *the right rank* > 1, and S_2 is derived by LB, RN or RM.

(3) The case where *the left rank* > 1, and S_1 is derived by LB, RN or RM.

Proof. (1) The case where *the left rank* $= 1$, *the right rank* $= 1$, and the outermost symbol of the *Mix formula* is $\Box$. In this case, we further divide the case into (A) the subcase where S_1 is derived by RN and S_2 is derived by RM, (B) the subcase where S_1 is derived by RN and S_2 is derived by LB, (C) the subcase where S_1 is derived by RM and S_2 is derived by RM, and (D) the subcase where S_1 is derived by RM and S_2 is derived by LB.

(A) When S_1 is derived by RN and S_2 is derived by RM, the bottom part of the proof is as follows:

$$\dfrac{\dfrac{\to A}{\to \Box A}\,\mathrm{RN} \qquad \dfrac{A \to B}{\Box A \to \Box B}\,\mathrm{RM}}{\to \Box B}\, Mix(\Box A)$$

We transform the above to the following:

$$\dfrac{\dfrac{\to A \quad A \to B}{\to B}\, Mix(A)}{\to \Box B}\,\mathrm{RN}$$

In the transformed figure, the *grade* of the *Mix* gets smaller. Hence, we can obtain a proof without *Mix* of the original endsequent.

(B) The case where S_1 is derived by RN and S_2 is derived by LB can be treated as in the proof of Theorem 4.8(1)(A).

(C) The case where S_1 is derived by RM and S_2 is derived by RM can be treated as in the proof of Theorem 6.5(1)(A).

(D) The case where S_1 is derived by RM and S_2 is derived by LB can be treated as in the proof of Theorem 6.5(1)(B).

(2) The case where *the right rank* > 1, and S_2 is derived by LB, RN or RM.

In this case, S_2 cannot be derived by RN nor by RM since *the right rank* > 1. The case where S_2 is derived by LB can be treated as in the proof of Theorem 2.8(2)(A).

(3) The case where *the left rank* > 1, and S_1 is derived by LB, RN or RM.

In this case, S_1 cannot be derived by RN nor by RM since *the left rank* > 1. The case where S_1 is derived by LB can be treated as in the proof of Theorem 2.8(3).

■

§8. Gentzen-type Formulation $\mathbf{KL}^*$ of KL

Now we reconstruct **KL** into a Gentzen-type formulation $\mathbf{KL}^*$ and prove the Cut elimination theorem for $\mathbf{KL}^*$.

Definition 8.1. The modal logic $\mathbf{KL}^*$ is constructed from **LK** by adding the modal operator $\Box$ and the rules of inference of the forms:

$$\text{LL}: \quad \frac{A, \Gamma \to \Delta, \Box B}{\Box A, \Gamma \to \Delta, \Box B}$$

and

$$\text{RK}: \quad \frac{\Sigma \to A}{\Box\Sigma \to \Box A}$$

that is, $\mathbf{KL}^* = \mathbf{LK} + \text{LL} + \text{RK}$.

It is well known that $\mathbf{K}=\mathbf{LK}+$ RK.

Next, we prove $\mathbf{KL} = \mathbf{KL}^*$ as logics.

(1) $\mathbf{KL} \supseteq \mathbf{KL}^*$.

For this, we only need to prove the following:

Lemma 8.2. *The inference* LL *is permissible in* **KL**.

Proof. We assume the provability of $\Box A \to A, \Box B$ in **KL** since it is trivial.

$$\cfrac{\cfrac{\Box A \to A, \Box B}{\Box A \to \Box B, A}\ \mathit{Interchange} \qquad A, \Gamma \to \Delta, \Box B}{\cfrac{\cfrac{\Box A, \Gamma \to \Box B, \Delta, \Box B}{(\mathit{Interchange})}}{\cfrac{\Box A, \Gamma \to \Delta, \Box B, \Box B}{\Box A, \Gamma \to \Delta, \Box B}\ \mathit{Contraction}}}\ \mathit{Cut}$$

■

(2) $\mathbf{KL}^* \supseteq \mathbf{KL}$.

Now, we prove the following lemma:

Lemma 8.3. *The axiom* L: $\Box A \supset (A \vee \Box B)$ *is provable in* $\mathbf{KL}^*$.

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A \to A, \Box B}\,\textit{Thinning}}{\Box A \to A, \Box B}\,\mathrm{LL}}{\Box A \to A, A \vee \Box B}\,\to\vee}{\Box A \to A \vee \Box B, A}\,\textit{Interchange}}{\Box A \to A \vee \Box B, A \vee \Box B}\,\to\vee}{\Box A \to A \vee \Box B}\,\textit{Contraction}}{\to \Box A \supset (A \vee \Box B)}\,\to\supset$$

■

So we have the following:

Theorem 8.4. $\mathbf{KL}$=$\mathbf{KL}^*$.

Now we prove the Cut elimination theorem for $\mathbf{KL}^*$.

Theorem 8.5. (Cut elimination) *Any* $\mathbf{KL}^*$ *proof can be transformed into a* $\mathbf{KL}^*$ *proof of the same endsequent without use of the inference Cut.*

As before, it is obvious that we only need to treat the following three cases:

(1) The case where *the left rank* $= 1$, *the right rank* $= 1$, and the outermost symbol of the *Mix formula* is $\Box$.

(2) The case where *the left rank* > 1, *the right rank* $= 1$, and S_1 is derived by LL or RK.

(3) The case where *the right rank* > 1, and S_2 is derived by LL or RK.

Proof. (1) The case where *the left rank* $= 1$, *the right rank* $= 1$, and the outermost symbol of the *Mix formula* is $\Box$.

In this case, we further divide the case into (A) the subcase where S_1 is derived by RK and S_2 is derived by RK, and (B) the subcase where S_1 is derived by RK and S_2 is derived by LL.

(A) When S_1 is derived by RK and S_2 is derived by RK, the bottom part of the proof is as follows:

$$\cfrac{\cfrac{\Gamma \to A}{\Box\Gamma \to \Box A}\,\text{RK} \qquad \cfrac{\Sigma \to B}{\Box\Sigma \to \Box B}\,\text{RK}}{\Box\Gamma, (\Box\Sigma)_{\Box A} \to \Box B}\,Mix(\Box A)$$

We transform the above to the following:

$$\cfrac{\cfrac{\Gamma \to A \quad \Sigma \to B}{\Gamma, \Sigma_A \to B}\,Mix(A)}{\Box\Gamma, (\Box\Sigma)_{\Box A} \to \Box B}\,\text{RK}$$

In the transformed figure, the *grade* of the *Mix* gets smaller . Hence, we can obtain a proof without *Mix* of the original endsequent.

(B) When S_1 is derived by RK and S_2 is derived by LL, the bottom part of the proof is as follows:

$$\cfrac{\cfrac{\Gamma \to A}{\Box\Gamma \to \Box A}\,\text{RK} \qquad \cfrac{A, \Sigma \to \Pi, \Box B}{\Box A, \Sigma \to \Pi, \Box B}\,\text{LL}}{\Box\Gamma, \Sigma \to \Pi, \Box B}\,Mix(\Box A)$$

where the *Mix formula* $\Box A$ does not appear in Σ, since the *the right rank* is only 1.
We transform the above to the following:

$$\cfrac{\cfrac{\cfrac{\cfrac{\Gamma \to A \quad A, \Sigma \to \Pi, \Box B}{\Gamma, \Sigma_A \to \Pi, \Box B}\,Mix(A)}{(Thinning, Interchange)}}{\Gamma, \Sigma \to \Pi, \Box B}}{\cfrac{(\text{LL}, Interchange)}{\Box\Gamma, \Sigma \to \Pi, \Box B}}$$

In the transformed figure, the *grade* of the *Mix* gets smaller. Hence, we can obtain a proof without *Mix* of the original endsequent.

(2) The case where *the left rank* > 1, *the right rank* $= 1$, and S_1 is derived by LL or RK.

In this case, S_1 cannot be derived by RK since *the left rank* > 1. So we treat only the case where S_1 is derived by LL. Now we divide the case into (A) the subcase where the *Mix formula* is $\Box B$ and (B) the subcase where the *Mix formula* is not $\Box B$.

(A) The case where the *Mix formula* is $\Box B$. Since *the right rank* $= 1$, S_2 is derived by LL or RK.

(a) When S_2 is derived by LL, the bottom part of the proof is as follows:

$$\dfrac{\dfrac{A,\Gamma \to \Delta, \Box B}{\Box A,\Gamma \to \Delta, \Box B}\,\text{LL} \qquad \dfrac{B,\Sigma \to \Pi, \Box C}{\Box B,\Sigma \to \Pi, \Box C}\,\text{LL}}{\Box A,\Gamma,\Sigma \to \Delta_{\Box B},\Pi,\Box C}\,Mix(\Box B)$$

where the *Mix formula* $\Box B$ does not appear in Σ, since the *the right rank* is only 1.
We transform the above to the following:

$$\dfrac{\dfrac{A,\Gamma \to \Delta, \Box B \quad \Box B,\Sigma \to \Pi, \Box C}{A,\Gamma,\Sigma \to \Delta_{\Box B},\Pi,\Box C}\,Mix(\Box B)}{\Box A,\Gamma,\Sigma \to \Delta_{\Box B},\Pi,\Box C}\,\text{LL}$$

In the transformed figure, the *rank* of the *Mix* gets smaller. Hence, we can obtain a proof without *Mix* of the original end-sequent.

(b) When S_2 is derived by RK, the bottom part of the proof is as follows:

$$\dfrac{\dfrac{A,\Gamma \to \Delta, \Box B}{\Box A,\Gamma \to \Delta, \Box B}\,\text{LL} \qquad \dfrac{\Sigma \to C}{\Box\Sigma \to \Box C}\,\text{RK}}{\Box A,\Gamma,(\Box\Sigma)_{\Box B} \to \Delta_{\Box B},\Box C}\,Mix(\Box B)$$

We transform the above to the following:

$$\dfrac{\dfrac{A,\Gamma \to \Delta, \Box B \quad \Box\Sigma \to \Box C}{A,\Gamma,(\Box\Sigma)_{\Box B} \to \Delta_{\Box B},\Box C}\,Mix(\Box B)}{\Box A,\Gamma,(\Box\Sigma)_{\Box B} \to \Delta_{\Box B},\Box C}\,\text{LL}$$

In the transformed figure, the *rank* of the *Mix* gets smaller. Hence, we can obtain a proof without *Mix* of the original end-sequent.

(B) When the *Mix formula* is not $\Box B$, the bottom part of the proof is as follows:

$$\dfrac{\dfrac{A,\Gamma \to \Delta, \Box B}{\Box A,\Gamma \to \Delta, \Box B}\,\text{LL} \qquad \Sigma \to \Pi}{\Box A,\Gamma,\Sigma_D \to \Delta_D,\Box B,\Pi}\,Mix(D)$$

We transform the above to the following:

$$\dfrac{\dfrac{\dfrac{\dfrac{A, \Gamma \to \Delta, \Box B \quad \Sigma \to \Pi}{A, \Gamma, \Sigma_D \to \Delta_D, \Box B, \Pi}\ Mix(D)}{A, \Gamma, \Sigma_D \to \Delta_D, \Pi, \Box B}\ (Interchange)}{\Box A, \Gamma, \Sigma_D \to \Delta_D, \Pi, \Box B}\ \mathrm{LL}}{\Box A, \Gamma, \Sigma_D \to \Delta_D, \Box B, \Pi}\ (Interchange)$$

In the transformed figure, the *rank* of the *Mix* gets smaller. Hence, we can obtain a proof without *Mix* of the original endsequent.

(3) The case where *the right rank* > 1 and S_2 is derived by LL or RK.

(A) The case where S_2 is derived by LL. Here, we divide the case into (a) the subcase where the *Mix formula* is $\Box A$ and (b) the subcase where the *Mix formula* is not $\Box A$.

(a) When the *Mix formula* is $\Box A$, the bottom part of the proof is as follows:

$$\dfrac{\Gamma \to \Delta \quad \dfrac{A, \Sigma \to \Pi, \Box B}{\Box A, \Sigma \to \Pi, \Box B}\ \mathrm{LL}}{\Gamma, \Sigma_{\Box A} \to \Delta_{\Box A}, \Pi, \Box B}\ Mix(\Box A)$$

where the *Mix formula* $\Box A$ appears in Σ, since *the right rank* > 1.

We transform the above to the following:

$$\dfrac{\dfrac{\Gamma \to \Delta \quad \dfrac{\dfrac{\dfrac{\Gamma \to \Delta \quad A, \Sigma \to \Pi, \Box B}{\Gamma, A, \Sigma_{\Box A} \to \Delta_{\Box A}, \Pi, \Box B}\ Mix_1(\Box A)}{A, \Gamma, \Sigma_{\Box A} \to \Delta_{\Box A}, \Pi, \Box B}\ (Interchange)}{\Box A, \Gamma, \Sigma_{\Box A} \to \Delta_{\Box A}, \Pi, \Box B}\ \mathrm{LL}}{\Gamma, \Gamma, \Sigma_{\Box A} \to \Delta_{\Box A}, \Delta_{\Box A}, \Pi, \Box B}\ Mix_2(\Box A)}{\Gamma, \Sigma_{\Box A} \to \Delta_{\Box A}, \Pi, \Box B}\ (Interchange, Contraction)$$

In the transformed figure, *Mix* is used twice. The *rank* of *Mix*$_1$ gets smaller than that of the original. So, *Mix*$_1$ can be eliminated. After the elimination of *Mix*$_1$, *Mix*$_2$ can be eliminated since the *the right rank* of *Mix*$_2$ is only 1, while the *the left rank* of *Mix*$_2$ is the same as in original. So, we can obtain a proof without *Mix* of the original endsequent.

(b) When the *Mix formula* is not $\Box A$, the bottom part of the proof is as follows:

$$\dfrac{\Gamma \to \Delta \qquad \dfrac{A, \Sigma \to \Pi, \Box B}{\Box A, \Sigma \to \Pi, \Box B}\,\mathrm{LL}}{\Gamma, \Box A, \Sigma_D \to \Delta_D, \Pi, \Box B}\,Mix(D)$$

We transform the above to the following:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \to \Delta \qquad A, \Sigma \to \Pi, \Box B}{\Gamma, (A)_D, \Sigma_D \to \Delta_D, \Pi, \Box B}\,Mix(D)}{(Thinning, Interchange)}}{\dfrac{A, \Gamma, \Sigma_D \to \Delta_D, \Pi, \Box B}{\Box A, \Gamma, \Sigma_D \to \Delta_D, \Pi, \Box B}\,\mathrm{LL}}}{\dfrac{(Interchange)}{\Gamma, \Box A, \Sigma_D \to \Delta_D, \Pi, \Box B}}$$

In the transformed figure, the *rank* of the *Mix* gets smaller. Hence, we can obtain a proof without *Mix* of the original endsequent.

(B) When S_2 is derived by RK, the bottom part of the proof is as follows:

$$\dfrac{\Gamma \to \Delta \qquad \dfrac{\Sigma \to A}{\Box\Sigma \to \Box A}\,\mathrm{RK}}{\Gamma, (\Box\Sigma)_D \to \Delta_D, \Box A}\,Mix(D)$$

where the *Mix formula* D appears in Σ, since *the right rank* > 1, and therefore $\Box D$ appears in $\Box\Sigma$.
We transform the above to the following:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Sigma \to A}{\Box\Sigma \to \Box A}\,\mathrm{RK}}{(Interchange)}}{D, \cdots, D, (\Box\Sigma)_D \to \Box A}}{(\mathrm{LL}, Interchange)}}{\dfrac{\Box D, \cdots, \Box D, (\Box\Sigma)_D \to \Box A}{(Contraction, Interchange)}}}{\dfrac{(\Box\Sigma)_D \to \Box A}{\dfrac{(Thinning, Interchange)}{\Gamma, (\Box\Sigma)_D \to \Delta_D, \Box A}}}$$

Hence we obtained a proof without *Mix* of the original endsequent.

■

§9. Gentzen-type Formulation RTA* of RTA

Now we reconstruct **RTA** into a Gentzen-type formulation **RTA*** and prove the Cut elimination theorem for **RTA***.

Definition 9.1. The modal logic **RTA*** is constructed from **LK** by adding the modal operator $\Box$ and the rules of inference LB and

$$\text{RTA}: \quad \frac{\Box B, \Sigma \to \Pi, A}{\Box B, \Sigma \to \Pi, \Box A}$$

that is, **RTA*** = **LK** + LB + RTA.

Next, we prove that **RTA**=**RTA*** as logics.

(1)**RTA**⊇**RTA***.

For this, we only need to prove the following two lemmas, the first one of which is immediate:

Lemma 9.2. *The inference* LB *is permissible in* **RTA**.

Lemma 9.3. *The inference* RTA *is permissible in* **RTA**.

Proof. We assume the provability of $\Box B, A \to \Box B\&A$ in **RTA** since it is almost trivial.

$$\cfrac{\Box B, \Gamma \to \Delta, A \qquad \cfrac{\cfrac{\Box B, A \to \Box B\&A \quad \Box B\&A \to \Box A}{\Box B, A \to \Box A}\,Cut}{A, \Box B \to \Box A}\,Interchange}{\cfrac{\Box B, \Gamma, \Box B \to \Delta, \Box A}{\Box B, \Gamma \to \Delta, \Box A}\,(Contraction, Interchange)}\,Cut$$

■

(2)**RTA*** ⊇**RTA**.

For this, we only need to prove the following three lemmas, the first and the third of which are immediate:

Lemma 9.4. *The axiom* T: $\Box A \supset A$ *is provable in* $\mathbf{RTA}^*$.

Lemma 9.5. *The axiom* A: $\Box A\&B \supset \Box B$ *is provable in* $\mathbf{RTA}^*$.

Proof.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \to B}{\Box A, B \to B}\,\textit{Thinning}}{\Box A, B \to \Box B}\,\mathrm{RTA}}{\Box A\&B, B \to \Box B}\,\& \to}{B, \Box A\&B \to \Box B}\,\textit{Interchange}}{\Box A\&B, \Box A\&B \to \Box B}\,\& \to}{\Box A\&B \to \Box B}\,\textit{Contraction}}{\to \Box A\&B \supset \Box B}\,\to\supset
$$

■

Lemma 9.6. *The inference* RR *is permissible in* $\mathbf{RTA}^*$.

So we have the following:

Theorem 9.7. $\mathbf{RTA}$=$\mathbf{RTA}^*$.

Now we prove the Cut elimination theorem for $\mathbf{RTA}^*$.

Theorem 9.8. (Cut elimination) *Any* $\mathbf{RTA}^*$ *proof can be transformed into an* $\mathbf{RTA}^*$ *proof of the same endsequent without use of the inference Cut.*

As before, it is obvious that we only need to treat the following three cases:

(1) The case where *the left rank* $= 1$, *the right rank* $= 1$, and the outermost symbol of the *Mix formula* is $\Box$.

(2) The case where *the right rank* > 1, *the left rank* $= 1$, and S_2 is derived by LB or RTA.

(3) The case where *the left rank* > 1, and S_1 is derived by LB or RTA.

Proof. (1) The case where *the left rank* $= 1$, *the right rank* $= 1$, and the outermost symbol of the *Mix formula* is $\Box$.

In this case, S_1 is derived by RTA and S_2 is derived by LB. So, the bottom part of the proof is as follows:

$$\dfrac{\dfrac{\Box A, \Gamma \to \Delta, B}{\Box A, \Gamma \to \Delta, \Box B}\,\text{RTA} \qquad \dfrac{B, \Sigma \to \Pi}{\Box B, \Sigma \to \Pi}\,\text{LB}}{\Box A, \Gamma, \Sigma \to \Delta, \Pi}\,Mix(\Box B)$$

where $\Box B$ does not appear in Δ and Σ, since *the left rank* $= 1$, *the right rank* $= 1$.
We transform the above to the following:

$$\dfrac{\dfrac{\Box A, \Gamma \to \Delta, B \quad B, \Sigma \to \Pi}{\Box A, \Gamma, \Sigma_B \to \Delta_B, \Pi}\,Mix(B)}{\dfrac{(Thinning, Interchange)}{\Box A, \Gamma, \Sigma \to \Delta, \Pi}}$$

In the transformed figure, the *grade* of the *Mix* gets smaller. Hence, we can obtain a proof without *Mix* of the original endsequent.

(2) The case where *the right rank* > 1, *the left rank* $= 1$, and S_2 is derived by LB or RTA.

(A) The case where S_2 is derived by LB. This case can be treated as in the proof of Theorem 2.8(2)(A).

(B) The case where S_2 is derived by RTA. Here, we divide the case into (a) the subcase where the *Mix formula* is $\Box A$ and (b) the subcase where the *Mix formula* is not $\Box A$.

(a) When the *Mix formula* is $\Box A$, the bottom part of the proof is as follows:

$$\dfrac{\dfrac{\Box C, \Gamma \to \Delta, A}{\Box C, \Gamma \to \Delta, \Box A}\,\text{RTA} \qquad \dfrac{\Box A, \Sigma \to \Pi, B}{\Box A, \Sigma \to \Pi, \Box B}\,\text{RTA}}{\Box C, \Gamma, \Sigma_{\Box A} \to \Delta, \Pi, \Box B}\,Mix(\Box A)$$

where S_1 is derived by RTA, since the outermost symbol of the *Mix formula* is $\Box$ and *the left rank* $= 1$.
We transform the above to the following:

$$\dfrac{\dfrac{\Box C, \Gamma \to \Delta, \Box A \quad \Box A, \Sigma \to \Pi, B}{\Box C, \Gamma, \Sigma_{\Box A} \to \Delta, \Pi, B}\,Mix(\Box A)}{\Box C, \Gamma, \Sigma_{\Box A} \to \Delta, \Pi, \Box B}\,\text{RTA}$$

In the transformed figure, the *rank* of the *Mix* gets smaller. Hence, we can obtain a proof without *Mix* of the original endsequent.

(b) When the *Mix formula* is not $\Box A$, the bottom part of the proof is as follows:

$$\dfrac{\Gamma \to \Delta \quad \dfrac{\Box A, \Sigma \to \Pi, B}{\Box A, \Sigma \to \Pi, \Box B}\text{RTA}}{\Gamma, \Box A, \Sigma_D \to \Delta_D, \Pi, \Box B} Mix(D)$$

We transform the above to the following:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \to \Delta \quad \Box A, \Sigma \to \Pi, B}{\Gamma, \Box A, \Sigma_D \to \Delta_D, \Pi, B} Mix(D)}{\Box A, \Gamma, \Sigma_D \to \Delta_D, \Pi, B}(Interchange)}{\Box A, \Gamma, \Sigma_D \to \Delta_D, \Pi, \Box B}\text{RTA}}{\Gamma, \Box A, \Sigma_D \to \Delta_D, \Pi, \Box B}(Interchange)$$

In the transformed figure, the *rank* of the *Mix* gets smaller. Hence, we can obtain a proof without *Mix* of the original end-sequent.

(3) The case where *the left rank* > 1, and S_1 is derived by LB or RTA.

(A) The case where S_1 is derived by LB can be treated as in the proof of Theorem 2.8(3).

(B) The case where S_1 is derived by RTA. Here, we divide the case into (a) the subcase where the *Mix formula* is $\Box B$ and (b) the subcase where the *Mix formula* is not $\Box B$.

(a) When the *Mix formula* is $\Box B$, the bottom part of the proof is as follows:

$$\dfrac{\dfrac{\Box A, \Gamma \to \Delta, B}{\Box A, \Gamma \to \Delta, \Box B}\text{RTA} \quad \Sigma \to \Pi}{\Box A, \Gamma, \Sigma_{\Box B} \to \Delta_{\Box B}, \Pi} Mix(\Box B)$$

We transform the above to the following:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box A, \Gamma \to \Delta, B \quad \Sigma \to \Pi}{\Box A, \Gamma, \Sigma_{\Box B} \to \Delta_{\Box B}, B, \Pi} Mix_1(\Box B)}{\Box A, \Gamma, \Sigma_{\Box B} \to \Delta_{\Box B}, \Pi, B}(Interchange)}{\Box A, \Gamma, \Sigma_{\Box B} \to \Delta_{\Box B}, \Pi, \Box B}\text{RTA} \quad \Sigma \to \Pi}{\Box A, \Gamma, \Sigma_{\Box B}, \Sigma_{\Box B} \to \Delta_{\Box B}, \Pi, \Pi} Mix_2(\Box B)}{\Box A, \Gamma, \Sigma_{\Box B} \to \Delta_{\Box B}, \Pi}(Contraction, Interchange)$$

In the transformed figure, *Mix* is used twice. The *rank* of Mix_1 gets smaller than that of the original. So, Mix_1 can be eliminated. After the elimination of Mix_1, Mix_2 can be eliminated since the *the left rank* of Mix_2 is only 1, while the *the right rank* of Mix_2 is the same as in original. So, we can obtain a proof without *Mix* of the original endsequent.

(b) When the *Mix formula* is not $\Box B$, the bottom part of the proof is as follows:

$$\dfrac{\dfrac{\Box A, \Gamma \to \Delta, B}{\Box A, \Gamma \to \Delta, \Box B}\text{RTA} \qquad \Sigma \to \Pi}{\Box A, \Gamma, \Sigma_D \to \Delta_D, \Box B, \Pi}\,Mix(D)$$

We transform the above to the following:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box A, \Gamma \to \Delta, B \quad \Sigma \to \Pi}{\Box A, \Gamma, \Sigma_D \to \Delta_D, (B)_D, \Pi}\,Mix(D)}{(Thinning, Interchange)}}{\Box A, \Gamma, \Sigma_D \to \Delta_D, \Pi, B}}{\Box A, \Gamma, \Sigma_D \to \Delta_D, \Pi, \Box B}\text{RTA}}{\dfrac{(Interchange)}{\Box A, \Gamma, \Sigma_D \to \Delta_D, \Box B, \Pi}}$$

In the transformed figure, the *rank* of the *Mix* gets smaller. Hence, we can obtain a proof without *Mix* of the original endsequent.

■

§10. Map of our modal logics

First, we introduce Gentzen-type formulation of **S4** and **KT** by citing from Ohnishi and Matsumoto [4].

Definition 10.1. (1) The modal logic **S4*** is constructed from **LK** by adding the modal operator $\Box$ and the rules of inference LB and

$$\text{S4}: \quad \dfrac{\Box\Sigma \to A}{\Box\Sigma \to \Box A}$$

that is, **S4*** = **LK**+ LB + S4.

(2) The modal logic **KT***, which is called as **M*** in [4], is constructed from **LK** by adding the modal operator $\Box$ and the rules of inference LB and RK, that is, **KT*** = **LK**+ LB + RK.

In [4], the following equalities are proved,

$$\mathbf{S4} = \mathbf{S4}^*$$

$$\mathbf{KT} = \mathbf{KT}^*$$

and the cut elimination theorem for $\mathbf{S4}^*$ and $\mathbf{KT}^*$ are proved.

Next, we investigate the relations among **RT4**, **MT4**, **MNT4**, **S4**, **RT**, **MT**, **MNT** and **KT**.

Theorem 10.2. $\mathbf{RT4} \supset \mathbf{MT4}$.[4]

Proof. First, $\mathbf{RT4} \supseteq \mathbf{MT4}$, since the inference RM is permissible in **RT4** and the axioms T and 4 are provable in **RT4**. While, by using the Cut elimination theorem for $\mathbf{RT4}^*$ and $\mathbf{MT4}^*$, it is easily proved that the formula of the form of the axiom K is provable in $\mathbf{RT4}^*$, but not in $\mathbf{MT4}^*$. Therefore $\mathbf{RT4} \neq \mathbf{MT4}$. ∎

Similarly, we can prove $\mathbf{MNT4} \supset \mathbf{MT4}$, $\mathbf{S4} \supset \mathbf{RT4}$, $\mathbf{S4} \supset \mathbf{MNT4}$, $\mathbf{RT} \supset \mathbf{MT}$, $\mathbf{MNT} \supset \mathbf{MT}$, $\mathbf{KT} \supset \mathbf{RT}$, $\mathbf{KT} \supset \mathbf{MNT}$, $\mathbf{MT4} \supset \mathbf{MT}$, $\mathbf{RT4} \supset \mathbf{RT}$, $\mathbf{MNT4} \supset \mathbf{MNT}$ and $\mathbf{S4} \supset \mathbf{KT}$, where the last relation is well known.

Theorem 10.3. $\mathbf{RT4} = \mathbf{MT4} + \mathrm{K}$.

Proof. First, it is easily proved that the inference RM is permissible in **RT4** and the axioms T, 4 and K are provable in **RT4**. Therefore $\mathbf{RT4} \supseteq \mathbf{MT4}+\mathrm{K}$. While the inference RR is permissible in $\mathbf{MT4}$+K and the axioms T and 4 are provable in $\mathbf{MT4}$+K. Therefore $\mathbf{RT4} \subseteq \mathbf{MT4} + \mathrm{K}$. ∎

Similarly, we can prove $\mathbf{S4} = \mathbf{MNT4}+\mathrm{K}$, $\mathbf{RT} = \mathbf{MT}+\mathrm{K}$, $\mathbf{KT} = \mathbf{MNT}+\mathrm{K}$, $\mathbf{S4} = \mathbf{RT4} + \mathrm{RN}$, $\mathbf{MNT4} = \mathbf{MT4} + \mathrm{RN}$, $\mathbf{KT} = \mathbf{RT} + \mathrm{RN}$, $\mathbf{MNT} = \mathbf{MT} + \mathrm{RN}$, $\mathbf{MT4} = \mathbf{MT} + 4$, $\mathbf{RT4} = \mathbf{RT} + 4$, $\mathbf{MNT4} = \mathbf{MNT} + 4$ and $\mathbf{S4} = \mathbf{KT} + 4$, where the last relation is well known.

So we obtained the Figure 1 in §1.

Finally, we investigate the location of **RTA**.

[4] Here, the symbol $\supset$ is used to mean the strict inclusion.

Theorem 10.4. **RTA** $\supset$ **RT4**.

This is immediate by using the Cut elimination theorem for **RTA*** and **RT4***.

Theorem 10.5. **RTA** *and* **S5** *are incomparable.*

Proof. It is easily proved that $\Box(A \supset A)$ is provable in **S5**, but it is not provable in **RTA**.

On the other hand by using the decision procedure for **S5** (see eg. Ohnishi and Matsumoto [5]), it is easily proved that A : $\Box A \& B \supset \Box B$ is not provable in **S5**. ∎

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